Mantra for Geometric Langlands

GLC is Categorical Harmonic Analysis
for D-modules on moduli space $\mathrm{Bun}_G(C)$
of G-bundles for a cpt Riemann surface, C

abelian, classical	non-abelian, categorical
$L^2(G) \cong L^2(\hat{G})$	$D(\underbrace{\mathrm{Bun}_G}_{\boxed{2}}) \simeq ?$
$G \circlearrowright L^2(G)$ translation operators	$\underbrace{\mathrm{Sat}_G}_{\boxed{3}} \circlearrowright D(\mathrm{Bun}_G)$
$\{e^{ixt}\}_{t \in \hat{G}}$ eigenbasis	$\{\star\}_{\star \in ?} \leftarrow \boxed{4}$ eigenbasis

$\boxed{2}$ Moduli of Bundles (1)
and Hitchin Fibration (2)

(1) 1. Line bundles

X variety

Picard variety of line bundles $\underline{\mathrm{Pic}(X)}$

$= H^1(X, \underbrace{\mathcal{O}_X^\times}_{\text{nowhere-vanishing functions on } X})$

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

$$\cdots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \overset{\overset{S}{\cap}}{H^1(X,\mathcal{O}_X^*)} \xrightarrow{c_1 = \deg} H^2(X,\mathbb{Z}) \to \cdots$$

$$\underline{\mathrm{Pic}^0}(X) = \ker(\deg : H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}))$$

$$= H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$$

<u>Ex</u> $X = C$ cpt. Riemann surface of genus g

$\underline{\mathrm{Pic}}^0(X) = \mathbb{C}^g/\mathbb{Z}^{2g}$ g-dimensional abelian variety

$\underline{\mathrm{Pic}}^0(X)$ is an abelian variety

$$A = \overset{\parallel}{H^1(X,\mathcal{O}_X)}/H^1(X,\mathbb{Z})$$

$A = V/\Lambda \leftrightarrow A^\vee = V^*/\Lambda^*$

$A^\vee = H^0(X,\Omega_X)^*/H^1(X,\mathbb{Z}) = \mathrm{Alb}(X)$ "Albanese variety"

Ω_X is sheaf of 1-forms

Fix $x_0 \in X$

$u : X \to \mathrm{Alb}(X)$ Albanese map

$$x \to \left(\int_{x_0}^{x} : \underset{\underset{H^0(X,\Omega_X)}{\cap}}{\omega} \mapsto \int_{x_0}^{x} \omega\right)$$

Ex 1/ $X = C$

AJ_{x_0} : $C \hookrightarrow \mathrm{Alb}(C) = \mathrm{Jac}(C)$ "Abel Jacobi"

$$\begin{tikzcd} H^1(C, \mathcal{O}_C) \arrow[r, phantom, "\cong"] \arrow[d, leftrightarrow] & H^0(C, \Omega_C)^* \\ H^1(C,\mathbb{Z}) \arrow[r, phantom, "\cong"] & H_1(C,\mathbb{Z}) \arrow[u, hook] \end{tikzcd} \Rightarrow \underline{\mathrm{Pic}}^0(C) \cong \mathrm{Jac}(C)$$

Rmk/

$$\begin{tikzcd} \pi_1(AJ_{x_0}): \pi_1(C) \arrow[r] \arrow[dr] & \pi_1(\mathrm{Jac}\, C) \\ & \pi_1^{ab}(C) = H_1(C,\mathbb{Z}) \arrow[u, hook] \end{tikzcd}$$

Note local system $\cong$ rep of π_1 $\cong$ flat connection

$$\begin{tikzcd} \text{Given: } \pi_1(C) \arrow[r] \arrow[dr] & \mathbb{C}^* \\ & \pi_1(C)^{ab} \arrow[u] \\ & \pi_1(\mathrm{Jac}(C)) \arrow[u, phantom, "\parallel"] \end{tikzcd}$$

rk 1 flat connection on C

We get rk 1 flat connection on $\mathrm{Jac}(C)$

$$\begin{tikzcd} D(\mathrm{Bun}_G(C)) \arrow[r, phantom, "\cong"] & QC(\mathrm{Flat}_G\, C) \\ \mathrm{Jac} \arrow[u] & \mathrm{Flat}_{GL_1} \arrow[u] \quad \text{skyscraper sheaf} \arrow[u, bend right] \end{tikzcd}$$

$G = GL_1$

D-module on Jac C

$A = \mathrm{Jac}\, C \quad A^\vee = \mathrm{Jac}\, C$

$$QC(A) \underset{FM}{\cong} QC(A^\vee)$$

Line bundle on A $\{ \mathcal{L} \leftrightarrow \mathcal{O}_{\mathcal{L}} \}$ skyscraper on $A^\vee = \underline{\mathrm{Pic}^0(A)}$

$$\mathrm{Jac} = \mathrm{Jac}\, C = H^0(C, \Omega_C)^* / H_1(C, \mathbb{Z})$$

$$\mathcal{A} = T^* \mathrm{Jac} = \mathrm{Jac} \times H^0(C, \Omega_C)$$

$$\begin{tikzcd}
\mathcal{A} = T^* \mathrm{Jac} = \mathrm{Jac} \times H^0(C, \Omega_C) \arrow[dr, "\text{fiber is Jac}"'] & & \mathcal{A}^\vee = \mathrm{Jac} \times H^0(C, \Omega_C) = T^* \mathrm{Jac} \arrow[dl] \\
& B = H^0(C, \Omega_C) &
\end{tikzcd}$$

$$QC(\mathcal{A}) \simeq QC(\mathcal{A}^\vee)$$

$$\dim T^* \mathrm{Jac} = \dim \mathrm{Jac} + \dim B$$
$$2g = g + g$$

$T^* \mathrm{Jac}$ is a symplectic space

Jac is Lagrangian

$$\mathbb{C}[B] = \mathbb{C}[T^* \mathrm{Jac}]$$

$\Rightarrow$ $T^* \mathrm{Jac} \to B$ is integrable system

<u>Def</u> (M^{2n}, ω)

$H \in \mathrm{Fun}(M)$ is completely integrable

if $\exists F_1 = H, F_2, \dots, F_n$ s.t.

- $dF_1 \wedge \cdots \wedge dF_n \neq 0$
- $\{F_i, F_j\} = 0 \quad \forall i, j$

Hitchin's integrable system

RmK (T-duality)

When one has a family of abelian varieties over B, one can construct

$A^\vee$ Fiberwise

$$A \overset{\text{T-dual}}{\longleftrightarrow} A^\vee$$

Moduli of bundles, flat bundles, Higgs bundles

$Bun_G = Bun_G(C)$ $\qquad$ $G \circlearrowright \begin{matrix} P \\ \downarrow \\ C \end{matrix}$

moduli space of G-bundles on C

$D(Bun_G)$ $\quad$ Note $D(X) \sim QC(T^*X)$

$\sim QC(T^* Bun_G)$

$T_P Bun_G(C) = H^{0,1}(C, \operatorname{ad} P)$

$G = GL_n$ $\qquad$ $A \in T_E \; Bun_n(C) = H^{0,1}(C, \operatorname{End} E)$

$$\bar\partial_E \rightsquigarrow \bar\partial_E + [A, \cdot]$$

$T_P^* Bun_G(C) = H^0(C, \Omega_C \otimes \operatorname{ad} P)$

$$T^* Bun_G(C) = \{ (P, \varphi) \mid \varphi \in H^0(C, \Omega_C \otimes \operatorname{ad} P) \}$$

$\quad \|$ $\qquad\qquad\qquad\quad \uparrow$

$Higgs_G(C)$ $\qquad$ Higgs field

$G = GL_1$

$E = \mathcal{L}$

$\mathrm{End}\, E = \mathrm{End}\, \mathcal{L} = \mathcal{L}^* \otimes \mathcal{L} = \mathcal{O}_C$

$\varphi \in H^0(C, \Omega_C \otimes_{\mathcal{O}_C} \mathcal{O}_C)$

$= H^0(C, \Omega_C) = B$

Flat C

$\nabla : E \to \Omega_C \otimes E$

$\nabla(fs) = df \otimes s + f \nabla s$

where $f \in \mathcal{O}_C$, $s \in E$

$(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 - \nabla_2)(s)$

$\nabla_1 - \nabla_2 \in H^0(C, \Omega_C \otimes \mathrm{End}\, E)$

Flat_G is an affine bundle modelled on

$T^* \mathrm{Bun}_G \to \mathrm{Bun}_G$

Betti moduli

<u>Rmk</u> $\mathrm{Flat}_G(C) \underset{\substack{\text{analytically} \\ \text{equivalent}}}{\sim} \mathrm{Loc}_G(C) = \mathrm{Hom}(\pi_1(C), G)/G$ character variety

de Rham moduli

but NOT algebraically

$$\begin{array}{ccc} T^* \mathrm{Bun}_G & & \mathrm{Flat}_G \\ & \searrow \; & \swarrow \; \text{not always} \\ & \mathrm{Bun}_G & \end{array}$$

Rmk 1 $\mathcal{M}_H \simeq (T^* \mathrm{Bun}_G)^{st}$

Hitchin moduli (soln to Hitchin eqn)

Hyper käler manifold I, J, K cpx structure

$(\mathcal{M}_H, I) \sim T^* \mathrm{Bun}_G$

$(\mathcal{M}_H, \text{any other}) \sim \mathrm{Flat}_{G_{\mathbb{C}}}$

(2) Hitchin System

1. Spectral Correspondence

Idea: Understand a linear map by its spectrum

V: cpx vector space $\dim_{\mathbb{C}} V = n$

IF $\varphi: V \to V$ is generic

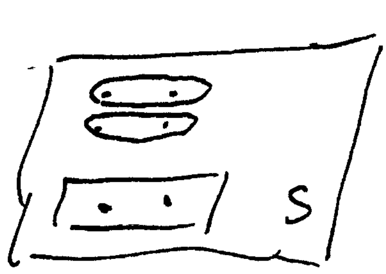
then φ has eigs $\lambda_1 \dots \lambda_n$ $\lambda_i \neq \lambda_j$ for $i \neq j$

$\lambda_i \rightsquigarrow L_i$ eigenspace

$V = L_1 \oplus \dots \oplus L_n$

Generalizations

① Introduce a parameter space



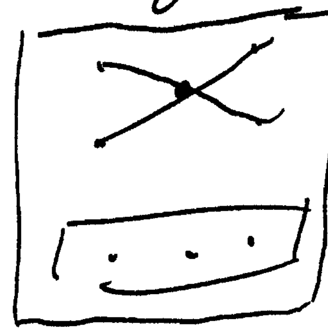
$\phi: S \to \mathrm{End}\, V$

$s \to \varphi_s$ is generic

$S \times \mathbb{C} \supset \bar{S} = \{(s, \lambda) \mid \lambda \text{ eig of } \varphi_s\}$

$$\begin{array}{ccc} \bar{S} & \mathcal{L}_{(s,\lambda)} \to \mathcal{L} \subset \bar{S} \times V \\ \downarrow n:1 \text{ cover} & \downarrow \qquad\quad \downarrow \\ S & (s, \lambda) \to s \end{array}$$

② allow φ_S to have repeated eigs



Note

Note: φ_s is as generic as possible

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} << \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} << \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

For now, use Jordan blocks with all 1s in the superdiagonal

$$\begin{matrix} \bar{S} \\ \downarrow \\ S \end{matrix} \text{ branched cover} \qquad \begin{matrix} \mathcal{L} \\ \downarrow \\ \bar{S} \end{matrix} \text{ line bundle}$$

$$\chi : \mathrm{End}\, V \to \mathbb{C}^n$$
$$\varphi \to (a_1(\varphi), \dots, a_n(\varphi))$$

where $\det(t \cdot id_V - \varphi) = t^n - a_1(\varphi) t^{n-1} + \dots + a_n(\varphi)$

$$\begin{matrix} \bar{\mathbb{C}}^n = \{(a_1, \dots, a_n, t) \in \mathbb{C}^n \times \mathbb{C} \\ \downarrow \\ \mathbb{C} \end{matrix}$$

Exer

$$\begin{matrix} \bar{S} & \to & \bar{\mathbb{C}}^n \\ \downarrow & & \downarrow \\ S & \xrightarrow{\chi \circ \varphi} & \mathbb{C}^n \end{matrix}$$

③ $GL(V) \circlearrowright \mathrm{End}(V)$

w/ $G \circlearrowright \mathfrak{g}$

④ Replace V by a vector bundle E on S

$\rightsquigarrow \phi \in H^0(S, \operatorname{End} E)$

⑤ Introduce coefficient object K

$$V \to K \otimes V$$

$\rightsquigarrow \varphi \in H^0(S, K \otimes \operatorname{End} E)$

K is rk r vector bundle on S

$S = X$ cpx alg. variety

On $U \subset X$, $K|_U \cong \mathbb{C}^{r} \otimes \mathcal{O}_U$

$\varphi|_U = (\varphi_1, \ldots, \varphi_r)$ $\quad \varphi_i \in H^0(U, \operatorname{End} E)$

spectral cover construction fails

<u>Defn</u> A K-valued Higgs field is $\varphi \in H^0(X, K \otimes \operatorname{End} E)$

s.t. $\phi \wedge \phi = 0 \in H^0(X, \overset{2}{\Lambda} K \otimes \operatorname{End} E)$

$\phi : E \to K \otimes E$

$\Leftrightarrow \phi^* : K^\vee \otimes E \to E$

$\Leftrightarrow \operatorname{Sym}^\bullet K^\vee \otimes E \to E$

$\qquad \underset{\mathcal{O}_Y}{\overset{\operatorname{Sym}^\bullet K^\vee}{\|}}$

$Y = \operatorname{tot}(K) \xrightarrow{\pi} X$

$\mathcal{O}(V) = \operatorname{Sym} V^*$

$\mathcal{E} \qquad\qquad E$

$$\begin{tikzcd} \pi_* : \left\{ \begin{array}{c} \text{quasi-coh sheaves} \\ \text{on } X \end{array} \right\} \arrow[r, "\sim"] & \left\{ \begin{array}{c} \text{quasi-coh} \\ \text{Higgs sheaves} \\ \text{on } X \end{array} \right\} \\ \left\{ \begin{array}{c} \text{coh sheaves on } Y \\ \text{wt finite support on } X \end{array} \right\} \arrow[u, hook] \arrow[r, "\sim"] & \left\{ \begin{array}{c} \text{coh. Higgs} \\ \text{sheaves on } X \end{array} \right\} \arrow[u, hook] \end{tikzcd}$$

Hitchin System

$S = X = C$

$K = \Omega_C$

$Y = \mathrm{tot}(K) = T^*C$

$$\left\{\begin{array}{c}\text{coh sheaves on } T^*C \\ \text{w/ support finite on } C\end{array}\right\} \overset{\pi_*}{\simeq} \left\{\begin{array}{c}\text{coh. Higgs} \\ \text{sheaves on } C\end{array}\right\}$$

$Y \to X$

$(E, \varphi) \in T^*\mathrm{Bun}_n \leftarrow GL(n)$ $\qquad \varphi_x : E_x \to T^*_x C \otimes E_x$

$\varphi : E \to \Omega_C \otimes E$

$\mathcal{O}_{T^*C} \curvearrowright \mathcal{E}$

$\mathrm{supp}\, \mathcal{E} = \Sigma \subset T^*C$

$= \{(x,\lambda) \mid \lambda \text{ an eig of } \varphi_x\}$

dim $V = n$	$\bar{S} \subset S \times \mathbb{C}$ $\downarrow n:1$ S	$\Sigma \subset T^*C$ $\downarrow n:1$ C	rk $E = n$
	$\mathcal{L}$ $\downarrow$ $\bar{S}$	$\mathcal{E}$ $\downarrow$ Σ	

$\chi : \mathrm{End}\, V \to \mathbb{C}^n$

$\chi_H : T^*\mathrm{Bun}_n \to \mathcal{B} \simeq \{\text{spectral curves}\}$

$$\begin{array}{ccc} (E,\varphi) & \mapsto & \mathrm{supp}\, \mathcal{E} = \Sigma \\ \text{is} & & \\ \mathcal{E} & & \end{array}$$

Given φ

$$\varphi \to \lambda^n - tr(\varphi)\lambda^{n-1} + \dots + (-1)^n \det(\varphi)$$

$$tr\,\varphi \in H^0(C, \Omega_C)$$

$$\det \varphi \in H^0(C, \Omega_C^{\otimes n})$$

$$\Rightarrow u \in \mathcal{B} = \bigoplus_{k=0}^{n} H^0(C, \Omega_C^{\otimes k})$$

$$\Sigma_u = \{(\lambda, t) / \lambda^n - u_1\lambda^{n-1} + \dots + (-1)^n u_n = 0\}$$

	abelian	non-abelian
duality	$T^*Jac \quad T^*Jac$ $\searrow \quad \swarrow$ $B = H^0(C, \Omega_C)$	$T^*Bun_n \supset \mathcal{A} \quad \mathcal{A}^\vee \subset T^*Bun_n$ $Pic\,\Sigma_u \to \mathcal{B}^{reg}$ $\downarrow u$ $\chi_H \searrow \mathcal{B} \swarrow \chi_H$ $\mathcal{B}$ ← "Hitchin Base"
n	$QC(T^*Jac)$	$QC(\mathcal{A}) \simeq QC(\mathcal{A}^\vee)$
	$\mathbb{C}[B] = \mathbb{C}[T^*Jac]$	$\mathbb{C}[\mathcal{B}] = \mathbb{C}[T^*Bun_n]$ Hitchin's system

$$\mathcal{B} \supset \mathcal{B}^{reg} = \{u \in \mathcal{B} / \Sigma_u \text{ smooth}\}$$

$$\begin{array}{ccc} T^*Bun_n & \to & \mathcal{B} \\ \cup & & \\ Pic\,\Sigma_u & \to & u \in \mathcal{B}^{reg} \end{array}$$

Rmk 1 This picture can be generalized to $G \Leftrightarrow \check{G}$

Donagi-Pantev

Rmk 1 $G = SL_2 \longrightarrow H^0(C, \Omega_C^{\otimes 2}) \Leftrightarrow \check{G} = PGL_2$

$GL_2 \to \bigoplus_{k=1}^{2} H^0(C, \Omega_C^{\otimes k})$

space of quadratic differentials

higher teichmuller theory

4d $N=2$ SUSY gauge thry

Rmk) $\Sigma \subset T^*C$

$\downarrow$

C ← Gaiotto curve (UV curve)

Seiberg witten (IR curve)

$\mathcal{B}$ coulomb branch of 4d-thry

from compactification of theory X along C

$\mathcal{M}_H$ coulomb branch of 3d-thry

from further compactification along S^1